David's Solutions to Homework Set 4, Cosmology, Fall 2002

1. The key is that in a flat universe, $\Omega_t = 1$ and never changes for all time.

$$\begin{aligned} \Omega_t &= \Omega_m + \Omega_r + \Omega_\Lambda && (1) \\ &= 1(\text{for } \textit{all} \text{ time}) && (2) \end{aligned}$$

Remember that $E = mc^2$ tells us that energy and matter can be interconverted between each other. So, light (radiation) has a mass equivalence which should be included in our tally of the density of the universe. When the universe is very hot (shortly after the big bang), the contribution due to radiation must be included, since the temperature (and thus energy) is so high. Thus, at first, $\Omega_m$ and $\Omega_r$ are comparable in magnitude. $\Omega_\Lambda$, on the other hand, is quite negligible, since during this time the critical density $\rho_c$ is very large. ($\Omega_\Lambda = \frac{\rho_\Lambda}{\rho_c}$ ... and $\rho_\Lambda$ is constant. So, if $\rho_c$ is very large, $\Omega_\Lambda$ is very small.)

As the universe expands, it cools. The energy density of radiation falls as the forth power of the scale factor ($\epsilon_r \sim \frac{1}{R^4}$). On the other hand, the matter density falls as the cube of the scale factor ($\rho_m \sim \frac{1}{R^3}$). So, $\Omega_r$ will decrease much faster than $\Omega_m$. However, since $\Omega_\Lambda$ is still very, very small and can't contribute significantly yet to $\Omega_t$, a decrease in $\Omega_r$ forces $\Omega_m$ to increase (since the sum $\Omega_t$ must always be 1). Thus, for a long time, $\Omega_m$ is very close to 1, while $\Omega_r$ diminishes.

Soon, however, $\rho_c$ becomes small enough so that $\Omega_\Lambda$'s contribution to $\Omega_t$ can't be ignored. ($\Omega_\Lambda = \frac{\rho_\Lambda}{\rho_c}$ ... and $\rho_\Lambda$ is constant. So, as $\rho_c$ gets smaller, $\Omega_\Lambda$ gets bigger.) In fact, pretty soon, $\Omega_\Lambda$ becomes comparable in magnitude to $\Omega_m$ and even overtakes it. Thus, $\Omega_m$ becomes smaller as $\Omega_\Lambda$ becomes larger.

For larger values of 1+z, the age of the universe is younger.

The farther away an object, the higher its 1+z by Hubble's Law. So, the light we receive today from more distant objects was originally emitted when the universe was younger. (More on that in Problem 4 below.)

Here is the *one* thing Prof. Ulmer wants you to remember 5 years after taking this course:

*** When we look at objects farther and farther away, we're looking at the universe earlier and earlier in time. ***

2. In a bottom-up model, the LOW mass clusters form first. The HIGH mass ones are expected to disappear first at higher 1+z.

However, we don't see what we expect ... we still see a disproprionately large number of high mass clusters compared to low mass clusters at higher 1+z. This suggests something might be wrong with .... i) our model of structure formation, ii) our ability to count low mass clusters, iii) our ability to count high mass clusters, or iv) all the above.

Bottom line: we've still got a LOT of work to do regarding understanding structure formation in the universe.

Punch line: GIVE US MORE MONEY ... *please*!!!

3. As the universe expands, it COOLS. See Figure 13.7, p.376 in your book.

4. Looking at an identical Earth 94 light years away, we would see events happening 94 years ago, which is the year 1908. The Cubs played the Detroit Tigers in the World Series in the fall of 1908 (and won ... the first time a team won back-to-back World Series, as in 1907, the Cubs won the World Series, also against the Tigers.)

For those needing more help seeing why looking at farther objects means looking back in time, consider this step-by-step explanation:

First, we must agree on two things:

a). The speed of light is finite. That is, it takes time for a light signal to travel from one point in space to another point. The speed of light (c) is about 3 x $10^8$ m/s in vacuum ... pretty fast, but still finite.

b). The fastest way to convey information is with light signals. For instance, the fastest way I can know you are standing in front of me is if I can SEE you.

Now, pretend you live on planet A, and your friend lives on planet B. The two planets are 1 light-year apart. (1 light-year = the distance light travels in one year.)

It's 2002, and you celebrate your 18th birthday, and you want your friend to watch your birthday party, so you start transmitting video images of your party to your friend while you are celebrating.

Question: When is the earliest time your friend will be able to watch you celebrating your 18th birthday party?

Answer: One year *later* in 2003, since it took the light signals 1 whole year to reach planet B. So, your friend thinks you're 18 years old when he sees your video, while in fact, the moment he's watching the video, you've already aged another year and are already 19 years old (2003). So, he's actually looking *back in time*. He's looking at something that happened on a planet quite far away from him, and it took time for light to travel the separation distance and transmit information to him.

What if your separation were *two* light-years? It would take *two* years for him to get the signal. How about 94 light-years? 94 years. How about 1 billion light years? 1 billion years! So, the farther the object is away from us, the farther back in time we're observing.

The only way we can know anything about a star or galaxy is by the light it emits. So, if it takes time for that light to reach us, due to the fact that the object is so far away, we must wait that long to receive the light. If the distance is 1 million light years, we must wait 1 million years to receive light which it emits right now. Or, the light which we receive right now from the object was first emitted 1 million years ago.

A little closer to home: If you stand 3 m away from me, it takes

$$\begin{aligned} \text{time} &= \frac{\text{distance}}{\text{velocity}} \quad (3) \\ &= \frac{3m}{3\text{x}10^8 m/s} \quad (4) \\ &= 1\text{x}10^{-8} s \quad (5) \end{aligned}$$

for the light from you to get to me. That's 10 nanoseconds ... awfully short, but still, it takes that little amount of time. So, in fact, I don't see you as you are at this instant, but as you were 10 nanoseconds ago!

Also, the Sun is 8 light-minutes away from us, and the moon is 1 light-second away. So, if the Sun suddenly stopped shining, we wouldn't know about it for 8 minutes, or if the moon blew up, we wouldn't know about it until a second afterward.

5. The radius of the observable universe is about $10^{28}$ cm.

To give you an appreciation for different size scales in nature, consider taking steps of roughly $10^6$ to $10^8$ (1 million to 100 million), (except for the first step, which is $10^5$). That is, every step we take is about a million to 100 million times smaller (or larger, depending on which way you're going) than the next size:

$10^{+28}$ cm = The Observable Universe
$10^{+23}$ cm = The Milky Way Galaxy
$10^{+15}$ cm = Our (Sol) Solar System
$10^{+09}$ cm = The Earth
$10^{+02}$ cm = Humans
$10^{-05}$ cm = Viruses (smallest living things)
$10^{-13}$ cm = Protons, Neutrons

So, in this ladder of scales, we can roughly compare the sizes of things. So, we can say, "Humans look as big to a virus as the Earth looks to humans." Or, "A person is as small compared to the observable universe as a proton is to a person." Or other such tongue-twisters.

Interestingly, things about the size of a human is roughly in-between the sizes of the smallest and largest things we know about.

Question: if the size of the *observable* universe is $10^{+28}$ cm, is that the size of the *actual* universe today? (Hint ... review question 4 above!)

6.

Given:

$$1 + z = 5\text{x}10^{60} \tag{6}$$

Since ...

$$1 + z = \frac{R_{now}}{R_{then}} \tag{7}$$

... then,

$$\frac{R_{now}}{R_{then}} = 5\text{x}10^{60} \tag{8}$$

7. A test of GR was to observe the PRECESSION of the major(or minor) axis of Mercury.

The orbits of the planets are not circular. They are ellipses. So, for each planet, there's a closest point (perihelion) and a farthest point (aphelion) in the orbit. However, these points do not stay fixed in space over time. There's a small shift per orbit. This shift is called the *precession* of the orbit. In Newtonian gravity, this precession can be explained by the gravitational influence of the other objects in the solar system, such as the other planets.

In the case of Mercury, the orbital precession is 574" (arcseconds, where 1 arcsecond = $\frac{1}{360}$ of a degree) per century. However, after very carefully tallying up the gravitational influence of all the other planets, Mercury's orbital precession should be only 531" per century. That is, Newtonian gravity could not account for 43" per century of Mercury's orbital precession. To account for this difference, some suggested that an undiscovered planet existed, or that Newton's gravitational law had to be slightly modified.

As it turns out, Einstein's GR could account for the missing 43" per century in Mercury's orbital precession. Since the Sun is so massive, GR predicts it curves space-time around it. Mercury's orbit is close enough to the Sun as to be affected by this space-time curvature. A GR calculation shows that Mercury's orbit would precess an additional 43" per century due to the space-time curvature, thus resolving the mystery of Mercury's orbital precession. This was a triumph for GR, since it could explain something another theory of gravity (Newton's) could not.

8. The deflection of the star light by the sun during a solar eclipse is evidenced by ... c) a change in the apparent position of the star relative to star images not close to the sun. This is an example of *gravitational lensing.*

See Figure 8.9, page 225 of your text.

9. k = 0 goes with $\Omega_0 = 1$.

10. The Friedman Equation ... a "Very Important Equation" in cosmology:

$$H_0^2 + \frac{kc^2}{R_0^2} = \frac{G8\pi\rho_0}{3} \tag{9}$$

$$H_0 \equiv \text{Hubble's Constant} \tag{10}$$

$$k \equiv \text{curvature constant} \tag{11}$$

$$\begin{aligned} c &\equiv \text{speed of light} & (12) \\ R_0 &\equiv \text{scale factor} & (13) \\ G &\equiv \text{gravitational constant} & (14) \\ \rho_0 &\equiv \text{density today} & (15) \end{aligned}$$

11.

$$\rho_c = \frac{3H_0^2}{8\pi G} \tag{16}$$

The critical density is the density necessary to stop the current expansion of the universe in the infinite future.

More precisely, the above statement says that the critical density is the density needed to make the kinetic energy (KE, energy due to an object's motion) go to zero when time (t) goes to infinity. We want the universe's expansion to *stop*, so we want its KE in the infinite future to go to *zero*. In this infinite future, the scale factor R goes to infinity, so the potential energy due to gravity (PE, energy due to an object's position in a gravitational field) also goes to zero.

*A Derivation of $\rho_c$*

You won't need the know the following for any exam, but I think it's simple and "beautiful" enough for you to regard it as an elegant example of the power of physics and very simple mathematics. I will be *very* gentle, and explain things line-by-line. Hang in there ... it's actually quite simple!

A basic principle in physics: Energy is *conserved.* That is, the *change* in the overall energy ($\Delta E$) is always zero. Mathematically, we can express the conservation of energy as:

$$\begin{aligned} \Delta E &= 0 \qquad (17) \\ E_{final} - E_{initial} &= 0 \qquad (18) \\ E_{final} &= E_{initial} \qquad (19) \end{aligned}$$

There are many forms of energy. We will consider only two:

1. The energy an object has due to its motion (v = velocity) is:

$$KE = \frac{1}{2}mv^2 \qquad (20)$$

2. The energy an object has due to its position in a gravitational field is:

$$PE = -\frac{GMm}{R} \qquad (21)$$

To find an object's *total* energy, and assuming the only two significant contributions to this energy are KE and PE, we simply add them up:

$$E = KE + PE \qquad (22)$$

So, putting this together with the conservation of energy, we can say:

$$\begin{aligned} \Delta E &= 0 \qquad (23) \\ E_{final} - E_{initial} &= 0 \qquad (24) \\ E_{final} &= E_{initial} \qquad (25) \\ E_{initial} &= E_{final} \qquad (26) \\ KE_{initial} + PE_{initial} &= KE_{final} + PE_{final} \qquad (27) \end{aligned}$$

So, to find the critical density, we want $KE_{final}$ and $PE_{final}$ to both be zero (since we want the expansion to stop in the infinite future). So ...

$$\begin{aligned}
KE_{initial} + PE_{initial} &= KE_{final} + PE_{final} &(28)\\
KE_{initial} + PE_{initial} &= 0 \qquad\quad + 0 &(29)\\
KE_{initial} &= -PE_{initial} &(30)
\end{aligned}$$

This is the equation which Prof. Ulmer started with ... and now you know where it came from. But he was a tad bit sloppy ... he neglected the minus sign. But he knew it drops out later, so he was in essence saving you some trouble. We'll see exactly now where the minus signs cancel out.

To finish the algebra, we simply substitute for KE and PE:

$$\begin{aligned}
KE_{initial} &= -PE_{initial} &(31)\\
\frac{1}{2}mv^2 &= -(-\frac{GM_cm}{R}) &(32)\\
M_c &= \frac{Rv^2}{2G} &(33)\\
\rho_c V &= \frac{R\dot{R}^2}{2G} &(34)\\
\rho_c &= \frac{R\dot{R}^2}{2GV} &(35)\\
\rho_c &= \frac{R\dot{R}^2}{2G(\frac{4}{3}\pi R^3)} &(36)\\
\rho_c &= \frac{3}{8\pi G}\frac{\dot{R}^2}{R^2} &(37)\\
\rho_c &= \frac{3}{8\pi G}H_0^2 &(38)
\end{aligned}$$

... which gives us the expression for the critical density.

The substitutions I made ...

... in line 34:

$$M_c = \rho_c V \tag{39}$$

where V is the volume, since by definition, $\rho = \frac{M}{V}$.

... in line 34:

$$v = \dot{R}. \tag{40}$$

... in line 36, I made the substitution for the volume of a sphere:

$$V = \frac{3}{4}\pi R^3. \tag{41}$$

... in line 38:

$$H_0 = \frac{\dot{R}}{R} \tag{42}$$

So, this demonstrates the power of physics and mathematics. We could convert a rather vague statement like "The critical density is what is needed to stop the expansion of the universe in the infinite future" into a very precise expression for the exact value of the critical density ... and relate it to something we can measure! And we only used a very simple concept ... the conservation of energy!

Another way to describe (with words) how $\rho_c$ relates to KE and PE is something like:

"The critical density is the density which provides enough gravitational potential energy to exactly balance the kinetic energy due to the expansion (motion) of the universe."

... or more succinctly using math, $KE = -PE$, which we derived from the conservation of energy above.

Also, remember that $\rho_c$ *changes in time*, depending on $H_0$. In the beginning, $\rho_c$ is very large, since the expansion rate is huge. But as the universe expands and slows down, $\rho_c$ gets smaller.

12.

$$\Omega_0 = \frac{\rho_0}{\rho_c} \tag{43}$$

This is simply a definition meant to simplify our concepts, as well as to confuse those who haven't taken a course in cosmology :)

13.

$$
\begin{aligned}
1+z &= \frac{\lambda_o}{\lambda_e} && (44)\\
&= \frac{800\text{nm}}{400\text{nm}} && (45)\\
&= 2 && (46)
\end{aligned}
$$

Also,

$$1+z=\frac{R_{today}}{R_{then}} \tag{47}$$

So,

$$
\begin{aligned}
\frac{R_{today}}{R_{then}} &= 2 && (48)\\
R_{then} &= \frac{1}{2}R_{today} && (49)
\end{aligned}
$$

So, the scale factor of the universe when the object emitted the light was 2 times *smaller* than today's value.

14.

This is *very* blue-shifted! When you do the calculation (which is simple), the object emitting this light is *approaching* at 3/5 the speed of light!

Incidentally, the Andromeda Galaxy (M31), part of our Local Group, is actually approaching us, (though much slower than 3/5 c!). There's some debate about whether it will actually plow into the Milky Way (our galaxy) in the future. Nothing to concern yourself about, though ... if it is headed for us, it won't reach us until long after you die (of hopefully less traumatic causes). Besides, there's so much space in-between stars of a galaxy that whole galaxies can pass through each other with very few actual star collisions. The gravitational changes during a galactic interaction will certainly affect the orbital motions of the individual stars in a galaxy, but the occupants of a planetary system around a star in one of the galaxies may not even realize another galaxy is passing through their own!

15. The Robertson-Walker metric:

$$(ds)^2 = (cdt)^2 - \left(\frac{R^2(t)(dr)^2}{1 - kr^2}\right) \tag{50}$$

This metric is the most general space-time metric for a *dynamic* (changing in time), *homogeneous* (all points are equal), and *isotropic* (all directions are equal) universe.

A note about "metrics":

Remember, a "metric" is simply the rule by which you determine distances between two points (or events) in space (or time). For instance, the Pythagorian theorem gives the metric in Euclidean (flat) space: $\Delta r^2 = \Delta x^2 + \Delta y^2 + \Delta z^2$. In flat space, this will always give you the distance between any two points, regardless how your initial coordinate system is set up. Someone in Berlin will agree with someone in New York that a meter stick is a meter long, even though they may disagree about where the meter stick is positioned (since they may be using very different points of reference.) We call the quantity $\Delta r^2$ (the square of) an "invariant interval" in space.

Likewise, in space-time, metrics describe the "distance" between two "points" in space-time. It is one dimension more complicated that just spatial geometry ... we're adding *time* into the equation. The "ds" is an interval in time and space. So, we call it a "space-time interval". (We use "d" instead of "$\Delta$" when the interval is infinitesimally small. That is, $\Delta s \rightarrow ds$ in the limit that $\Delta s$ goes to zero).

Compare Figure 6.3 (p. 155), and Figure 7.5 (p. 188) in your text.

When ds = 0, the resulting relationship describes the equation for a "null geodesic", the path in space-time which a photon would take. For instance, in flat space, k = 0, and given R = 1 (today), you can solve (simple algebra ... try it!) the Robertson-Walker metric to give you:

$$\left(\frac{dr}{dt}\right)_{photon} = c \tag{51}$$

... that is ...

$$v_{photon} = c \tag{52}$$

... that is, the speed of a photon (light) is c. (But of course!)

<u>Tying some ideas together:</u>

The *Robertson-Walker metric* describes the *geometry* of the *space-time* of a *dynamic, homogeneous, isotropic* universe. It tells us how to measure "distances" in space time. It accounts for the changing nature of the universe by using a scale factor R(t) which changes in time. It also allows for different possible geometries by incorporating a curvature constant k.

The *Friedman Equation* describes the *physics*, based on the *conservation of energy.* It shows us *how the scale factor R(t) changes with time.*

The Robertson-Walker metric and Friedman Equation are two of the *mathematical and physical tools* which allow us to *model* the universe and *scientifically* ask our big cosmological questions, once confined to mere philosophical/religious speculation.

From these equations, we can derive important quantities, such as the Hubble constant, which we can *observationally measure.* How well the *theoretical* and *observational* values for these quantities agree tells us whether our ideas about the universe are correct.