David's Solutions to Homework Set 5, Cosmology, Fall 2002

1. The two basic ingredients we need to derive the de-acceleration equation from the escape equation (Friedman equation):

i) The First Law of Thermodynamics:

$$dU = dQ - pdV \tag{1}$$

... where

$$\begin{aligned} dU &= \text{change in internal energy} \quad (2) \\ dQ &= \text{change in heat} \quad (3) \\ p &= \text{pressure} \quad (4) \\ dV &= \text{change in volume} \quad (5) \end{aligned}$$

ii) Adiabatic Condition:

$$dQ = 0 \tag{6}$$

Thus, using the adiabatic condition in the First Law of Thermodynamics simplifies Eq.(1) to be:

$$dU = -pdV \tag{7}$$

(For an explanation of the meaning behind these equations, please see Appendix A.)

2. The equation which tells us that negative pressure corresponds to energetically favorable expansion of the universe is:

$$dU = -pdV \tag{8}$$

If p is negative, then the two negative signs will cancel. Furthermore, since the change in volume is positive, *expansion* of the universe corresponds to a *positive* value of dV. So, you'll have:

$$dU = \text{a positive number} \tag{9}$$

Anything that *increases* the energy of the universe is good! A good analogy for energy is *money* (see Appendix A). Any system will tend toward a state where the energy is maximum. Likewise with money: More money is good; most people will tend toward a state of maximum money.

So, given $dU = -pdV$, the universe will *want* to expand if the pressure is negative, because then the change in energy is positive.

Conversely, if the pressure is *positive*, then when the universe expands (dV is positive), dU is a *negative* quantity. That is, the universe is *losing* energy as it expands. Money analogy: will you stay in business very long if you keep losing money? Probably not! Likewise, the universe wants to *slow down* its expansion if it is losing energy, that is, when pressure is positive.

Admittedly, a *positive* pressure causing the universe to *slow down* its expansion and a *negative* pressure causing the universe to *speed up* its expansion seems somewhat counter-intuitive. When we talk about pressure in everyday life, we have the picture of positive pressure pushing outward against the sides of an object, pushing it apart, that is, causing it to speed up its expansion. But the key difference is that in everyday life, positive pressure causes an outward force only because there's a *difference* in pressure; the pressure inside the object is greater than the pressure outside the object ... hence, the object expands. However, in a homogeneous and isotropic universe, the pressure everywhere is the *same*. And since there's nothing "outside the universe", we can't define a pressure beyond the boundaries of our universe. So, we can't use our everyday experience with pressure on a universal scale.

Rather, think of pressure on a universal scale as being caused by the *energy density due to radiation*. And since $E = mc^2$, radiation's energy density is equivalent to a matter density. And what does matter density cause? It is the source of *gravity*. Just as the earth (a big ball of matter and thus a source of gravity) slows down an object I throw up into the air and causes it to fall back down, so the matter (energy) density in the universe slows down its expansion. Thus, a *positive* pressure corresponds ultimately to a force which wants to *slow down* the expansion of the universe. Conversely, a *negative* pressure corresponds to a force which acts in the opposite direction, one which will *speed up* the expansion.

We know that whatever is causing our universe to accelerate can't be normal radiation pressure (which is positive, and would thus slow down the universe). Rather, it is something we don't know about right now. That's why we call it "dark" energy. A leading candidate for this dark energy is the energy in the vacuum due to quantum mechanical effects, but that's a story for another time.

3. What Prof. Ulmer wants you to know about this problem:

We can derive the Hubble Relation with just the Robertson-Walker metric. We don't need to use the Friedman (escape) equation to derive it.

(For more mathematical details, please see Appendix B.)

4. What Prof. Ulmer wants you to know about this problem:

* Einstein assumed the universe was *static* and *adiabatic*.
* Einstein's theory of general relativity did not have any static solutions, though. It said the universe had to be dynamic, either expanding or contracting. Not fully trusting his own theory, Einstein had to force his equations to allow for a static universe.
* A *static* universe requires that the pull of gravity is exactly counterbalanced by a repulsive force, which he symbolized using the cosmological constant (Λ).
* Einstein called this cosmological constant (Λ) his "greatest blunder", because Hubble soon discovered that the universe was actually *expanding* and *not* static. Einstein had been in a position to make the greatest prediction of his theory of general relativity, namely, that the universe was dynamically changing (expanding or contracting but not static), but he didn't trust his own theory completely. That's why he put in the cosmological constant as a "fudge factor".
* We now know that the universe is actually *accelerating*, and so Einstein's original idea of a cosmological constant may be quite useful after all! Indeed, rather than his greatest blunder, Λ could be one of Einstein's greatest realizations. Unfortunately, he didn't live long enough to appreciate this himself.

(For more mathematical details, please see Appendix C.)

5. When $\Omega_\Lambda = 0$,

$$\Omega_m = 2q \tag{10}$$

or,

$$q = \frac{1}{2}\Omega_m \tag{11}$$

When Ω_Λ is not zero,

$$q = \frac{1}{2}\Omega_m - \Omega_\Lambda \tag{12}$$

For a flat universe,

$$\begin{aligned} \Omega_t &= \Omega_m + \Omega_\Lambda && (13) \\ &= 1 && (14) \end{aligned}$$

For an empty universe,

$$\Omega_m = 0 \tag{15}$$

Thus, for a flat, empty universe,

$$\begin{aligned} \Omega_t &= \Omega_m + \Omega_\Lambda && (16) \\ &= 0 + \Omega_\Lambda && (17) \\ &= 1 && (18) \end{aligned}$$

So,

$$\Omega_\Lambda = 1 \tag{19}$$

Finally, substituting $\Omega_m = 0$ and $\Omega_\Lambda = 1$ into Eq.(12), we find for a flat, empty universe:

$$\begin{aligned} q &= \frac{1}{2}\Omega_m - \Omega_\Lambda && (20) \\ &= \frac{1}{2}(0) - (1) && (21) \\ &= -1 && (22) \end{aligned}$$

For our universe, $\Omega_m = 0.3$ and $\Omega_\Lambda = 0.7$ so, Eq.(12) becomes:

$$\begin{aligned} q &= \frac{1}{2}\Omega_m - \Omega_\Lambda && (23) \\ &= \frac{1}{2}(0.3) - (0.7) && (24) \\ &= -\frac{11}{20} && (25) \end{aligned}$$

This is *negative*, so our universe is *accelerating*.

IMPORTANT: q > 0 means *de-accelerating* (slowing down), and q < 0 means *accelerating* (speeding up).

6. Going back in time, regardless of what the geometry of the universe is today, the original geometry of the universe tends toward a *flat* universe, with $\Omega_t = 1$.

(For the simple mathematical proof, see Appendix C.)

7. On the R vs. t plot ...

a. $\dot{R} = 0$ is a flat, horizontal line.
b. R = ct is a line with a slope of c. (Figure 10.9 (b), p.283 in your text)
c. A closed universe w/ $\Lambda = 0$. (Figure 11.8, p. 311, dashed line)
d. A Lemaitre universe. (Figure 11.8, p. 311, solid line)

Suggestion: Gain practice drawing R vs. t plots for the different cosmological models listed in Table 11.2, p.312 of your text. Know these models by name, too, since Prof. Ulmer mentioned all of them at one time or another.

8. In a Lemaitre universe (or any closed universe, for that matter), we should find mirror images of objects 180 degrees away from the line of sight of that object.

The idea is simply that in a closed universe, the geometry is spherical, and so a light ray will eventually come back to its source. This is basically *gravitational lensing* applied to the universe as a whole. For example, if you throw a ball fast enough on the earth, it will fly all the way around the globe and hit you on the back of your head! Likewise, if you shine a beam of light in a closed universe, it will eventually come back to you. So, if you see the image of an object in one direction, you should be able to turn around and see another image (which has traveled around the whole universe on its way back to the source) of the same object! Of course, this assumes that there has been enough time for the light to travel all the way around the universe to reach you from the other direction.

Also, the ages of the two images may not be the same, since the path length in one direction may be different than in the other direction. For example, I could go to New York by flying east (smart) or west (not so smart). The east journey will be relatively short, whereas the west journey requires I go all the way around the globe and reach New York from the other direction! Remember, in space-time, the farther the object, the farther back in time we're looking. So, if we see double images of an object, and one image traveled a much longer distance to reach us, then that image will be further back in time than the other image. Thus, we'd be able to see the same object at different times in its evolution!

9. *Entropy* is a measure of *disorder* in a system. Systems tend toward states of more entropy (more disorder). This, in fact, is the Second Law of Thermodynamics: On a universal scale, entropy always increases (or at best, stays the same).

For instance, your room naturally gets more messy, not more orderly. Or, if a glass pitcher falls on the floor, it shatters into a zillion pieces. Those pieces won't spontaneously come back together to form a whole pitcher again, even though there's no physical reason why it can't. Or, if I open a bottle of fragrance, soon, the whole room smells nice because the molecules of fragrance have dispersed into the greater volume of the room. They won't spontaneously flow back into the bottle, even though nothing's prohibiting them from doing so.

That things tend toward greater disorder (increased entropy) may be linked to our perception of the direction in which time seems to flow. We have a very definite sense of past, present, and future. We don't wake up one morning to find it is *yesterday*, although, that might be nice, assuming we retained knowledge of what happened during the next day, like a midterm exam :) . If we watched a movie and saw a zillion glass pieces on the floor spontaneously fly together and became a glass pitcher on the kitchen counter, we would all know that the movie was being played backward. Our sense of the direction of time (or the *arrow of time*) seems obvious.

Is increasing entropy related to our expanding universe? If so, then if the universe begins to collapse, will entropy begin to *decrease*? Will the arrow of time reverse itself and start pointing backward? Will we all begin our lives as seniors, "grow" younger, and end our lives back in the womb?

If entropy does decrease during collapse, then when the Big Crunch occurs, the universe will be in a highly ordered, low entropy state. Thus, the universe may "bounce" and re-create itself. Everything will flow forward again, and entropy will start to increase.

On the other hand, decreasing entropy seems to defy common sense. It certainly may have nothing to do with whether the universe is expanding or contracting. Collapsing the walls of a room in which I opened a bottle of fragrance does not necessarily mean that the molecules of fragrance will start to flow back into the original bottle. In fact, they most likely will not! So, even if the universe collapses, entropy may still increase. If so, then when the Big Crunch occurs, there may *not* be a bounce since it will be in a state of high disorder, high entropy. We will have then essentially used up all our "order", leaving no more to re-create a new universe.

The behavior of *energy density* is very different than that of *entropy*. Energy is the capacity to do work. If I measure an amount of energy E in a volume V, then the energy density is $\epsilon = \frac{E}{V}$. The conservation of energy says the total energy E is constant. Thus, as the volume V increases, the energy density ϵ *decreases*. Hence, as the universe expands (V gets bigger), ϵ *decreases*. If the universe collapses (V gets smaller), ϵ *increases*. Not so with entropy. Entropy always *increases*.

Footnote on entropy: If entropy always increases, then how can we grow or heal or get stronger or become smarter? Answer: Yes, in these cases, entropy *decreases* locally, but only at the cost of *increasing* the overall entropy even more. That is, growing or getting smarter, etc., costs a lot of energy. The only way to grow and get smarter is to eat food, convert the food into energy to drive our cellular processes, and run around campus rushing to your next class. Along the way, a lot of heat is produced, enough to keep your core temperature at an average of 98°F. This heat increases the total entropy of the universe, even though in your specific person, the local value of the entropy decreased somewhat. As another example, we certainly can glue back together a zillion pieces of glass to re-form the pitcher, but that would require a lot of effort and energy. While the entropy of the glass pitcher may in the end be less than its zillion constituent pieces, the entropy of the universe increased due to your gluing effort.

10. The Hubble relation is:

$$v = HD \tag{26}$$

... where

$$\begin{aligned} v &= \text{velocity} && (27) \\ H &= \text{Hubble constant} && (28) \\ D &= \text{physical distance to the object} && (29) \end{aligned}$$

Dimensionally, we can see that the Hubble constant has units of $\left(\frac{1}{\text{time}}\right)$:

$$\begin{aligned} H &= \frac{v}{D} && (30) \\ &= \frac{cm/s}{cm} && (31) \\ &= \frac{1}{s} && (32) \end{aligned}$$

Thus, to get something that has units of time from H, we simply invert it:

$$\begin{aligned} t_H &= \frac{1}{H} && (33) \\ &= \frac{1}{\left(\frac{1}{s}\right)} && (34) \\ &= s && (35) \end{aligned}$$

... and t_H is called the *Hubble time.*

The Hubble time t_H represents the age of the universe if it was always expanding at the same rate. That is, t_H is the age of a *null universe* which has no matter ($\Omega_m = 0$) and no cosmological constant ($\Omega_\Lambda = 0$). A null universe expands at a constant rate, because there's no source of gravity to slow down the expansion, and no repulsive force to speed up the expansion.

Imagine you are driving a car at a constant speed, say, 10 mph. Then, if you lose track of time but notice that you just arrived at a town you knew was exactly 30 miles from your starting position, you could easily figure out how long you were traveling. Since $v = \frac{D}{t}$ (think of the units of speed), the time of travel would be:

$$\begin{aligned} \text{time of travel} &= \frac{D}{v} && (36) \\ &= \frac{30 \text{ miles}}{10 \text{ miles/ hr}} && (37) \\ &= 3 \text{ hours} && (38) \end{aligned}$$

In the same way, we can figure out the age of the universe from the Hubble constant, since velocity increases proportionally with distance exactly as the Hubble constant ($v = HD$). That is,

$$\begin{aligned} t_H &= \frac{D}{v} && (39) \\ &= \frac{D}{HD} && (40) \\ &= \frac{1}{H} && (41) \end{aligned}$$

Notice that this holds true for any object in the universe. By Hubble's Relation, if D gets bigger, so does v, and you always get the same ratio, which is t_H, the maximum possible age of the universe.

What about a realistic universe, that is, where Ω_m is positive? Well, introduce any matter into the universe and the expansion will want to *slow down* due to the attractive gravitational force. The more matter (the greater the Ω_m), the more the expansion will tend to slow down. So, universes with greater Ω_m are *younger* than their current value of the Hubble constant would seem to suggest. Why? Because the Hubble constant is ever decreasing in a de-accelerating universe. The analogy with the car might help again.

Say you start driving your car at 10 mph like before, but now you're gradually slowing down. By the time you reach that town exactly 30 miles away, you look at your speedometer and it reads 5 mph. Well, you do the math, and you figure you were traveling for ...

$$\text{time of travel} = \frac{D}{v} \tag{42}$$

$$= \frac{30 \text{ miles}}{5 \text{ miles/ hr}} \tag{43}$$

$$= 6 \text{ hours} \tag{44}$$

But that can't be right, because earlier in your journey, you were traveling *faster* than 5 mph. So, you must have gotten to the town *sooner* (less time) than your above calculation would suggest. Thus, your estimated time of travel of 6 hours (based on the assumption that you were always traveling at 5 mph) is an *over*-estimate. In fact, your time of travel was *less*.

Likewise, the more the universe de-accelerates (slows down), the less is its age compared to the Hubble time $t_H = \frac{1}{H}$.

Thus, the *greater* Ω_m (more mass ... more gravity ... more de-acceleration), the *younger* its age compared to the Hubble time $t_H = \frac{1}{H}$.

As it turns out, the following is true:

Ω_m	Actual age of this universe
0	$t_{age} = t_H$
<1	$\frac{2}{3}t_H < t_{age} < t_H$
= 1	$t_{age} = \frac{2}{3}t_H$
>1	$t_{age} < \frac{2}{3}t_H$

$t_H = \frac{1}{H}$

For a nice graphical representation of the above, see Figure 11.5, p. 308 in your text.

The key thing to remember: the *greater* the Ω_m, the *younger* is the age of the universe compared with the Hubble time $t_H = \frac{1}{H}$.

Appendix A

More On The First Law of Thermodynamics and The Adiabatic Condition

The meaning behind the equations:

Math is really the ultimate shorthand, the most precise "language" which humans have invented (or, as some would insist, *discovered*). I joke that physicists and mathematicians are ultimately the most lazy people in the world, but the point behind the joke is that we can say a LOT with just a few symbols. Rather than writing a hundred words, we write a few symbols. You'll see what I mean in the following:

The First Law of Thermodynamics says:

$$dU = dQ - pdV \tag{45}$$

What does this equation mean? It is simply another way of expressing a fundamental principle in physics, the *conservation of energy.* We used the conservation of energy in Homework Set 4, Problem 11, when we derived the critical density. The Friedman equation (which Prof. Ulmer calls the "escape equation") is also based on the conservation of energy. And now, the First Law of Thermodynamics is yet another form of the same principle.

A good analogy for "energy" is "money". The conservation of energy is just like the conservation of money. That is, if you give me \$20, you expect to receive \$20 worth of goods and/or services. If you give me \$20 to purchase something that I'm selling for \$10, you would expect \$10 in change. What you give me is what I should give back. Money is *conserved.* Likewise for energy. If I start with a certain amount of energy, I expect all of it to be there in the end. It may be converted to different forms of energy, but when I count it all up, I will be able to account for every "cent". And just like a good accountant who worries about a missing cent, a good physicist worries if there's some "missing energy" left over that isn't accounted for.

To see how the First Law of Thermodynamics describes the conservation of energy, let's re-write Eq.(1) and express it in terms of dQ:

$$dQ = dU + pdV \tag{46}$$
$$dQ = dU + dW \tag{47}$$

... where the quantity pdV is the amount of *work* done by this particular system on its surroundings. That is,

$$dW = pdV \tag{48}$$

Now we are in a position to realize what the First Law of Thermodynamics really says. Eq.(47) says when a certain amount of heat (dQ) is added to a system, it will:

i) change the system's internal energy (dU), (for example, by increasing its temperature);
ii) enable the system to do work (dW), (for example, by increasing the volume and so push something, like a piston); or ...
iii) a bit of both the above.

Another way to say it ... "Energy is conserved." It is not created or destroyed. One form of energy (heat dQ) can be transformed (=) into other forms (internal energy dU and/or work dW). It can be transferred to other systems, (for instance, when the work done by the system affects the surroundings or another system). But it is always conserved.

(So ... it took me more than a page to describe the simple statement $dU = dQ - pdV$. Math is *such* a timesaver!!)

Onward to the second statement: $dQ = 0$. This is the *adiabatic condition.* We encountered the word *adiabatic* when we were considering the cosmic microwave background (CMB) and the fluctuations in it. The word originates from the Greek word *adiabatos*, which means "not passable". In the current context, you can interpret the adiabatic condition as saying:

"Heat does not pass into or out from the system."

That is, "The net change in heat is zero."

Or even more succinctly,

$$dQ = 0 \tag{49}$$

Since there's nothing "outside" the universe as a whole, heat does not "pass into or out from" it. Thus, for the universe as a whole, $dQ = 0$ is certainly a reasonable assumption. By assuming adiabatic conditions, we can simplify the First Law of Thermodynamics to yield $dU = -pdV$. Using this simplified form of the First Law, we can derive the *fluid equation.* Then, using the fluid equation with the Friedman (escape) equation, we can derive the *acceleration equation*, and identify the de-acceleration parameter,

$$q = -\ddot{R}R/\dot{R}^2 \tag{50}$$

(Don't worry about the *fluid equation* and the *acceleration equation*, since Prof. Ulmer didn't present them.)

Incidentally, in the context of fluctuations in the CMB, the word *adiabatic* meant that radiation (light) could *not pass through* the matter, and so radiation "followed" the matter. So, in this theory, the fluctuations in temperature revealed the fluctuations in matter density, and if we expect the fluctuations to be 1 degree, and we actually do see them to be 1 degree, then we know we live in a flat universe (yadda yadda yadda ...).

Appendix B

Hubble's Law from the Robertson-Walker Metric

There are two ways to do this problem.

1. Dimensional Method:

The first method uses dimensional arguments, but is somewhat sloppy, since it makes certain assumptions which are questionable. For instance, it assumes the space-time interval is zero, which is only true for a photon. This is the method which Prof. Ulmer presented:

The Robertson-Walker metric:

$$(d\tau)^2 = (cdt)^2 - \left(\frac{R^2(t)(dr)^2}{1 - kr^2}\right) \tag{51}$$

Do the following:

* Set $d\tau = 0$ (even though this is true only for a photon):
* Set k = 0 (which is a reasonable assumption since we're looking at regions close to us, and all geometries appear flat locally)
* Change dt to t (which is dimensionally ok to do)
* Change dr to r (which is dimensionally ok to do)
* Change c to v (which is dimensionally ok to do, but questionable in principle)

With all these changes, and after taking a square-root, you will get:

$$\begin{aligned}
vt &= R(t)r && (52)\\
v &= \frac{R(t)}{t}r && (53)\\
v &= \dot{R}(t)r && (54)\\
v &= \dot{R}(t)\frac{D}{R(t)} && (55)\\
v &= \frac{\dot{R}(t)}{R(t)}D && (56)\\
v &= HD && (57)
\end{aligned}$$

... where we used D = R(t)r, (D is the physical distance, R(t) is the scale factor, and r is the unitless co-moving coordinate).

2. Correct Method:

This method is rigorously correct and is the proper way to derive the Hubble Relation from the Robertson-Walker method. However, it requires some elementary differential and integral calculus.

An equivalent form of the Robertson-Walker metric is:

$$(d\tau)^2 = -(cdt)^2 + \left(\frac{R^2(t)(dr)^2}{1-kr^2}\right) \tag{58}$$

To find a proper distance D (the physical distance between two objects at the same instant in proper time), set dt = 0. And as before, we set k = 0, because we're concerned about a local relationship.

After taking the square-root, we find that the space-time interval is just a space interval:

$$d\tau = R(t)dr \tag{59}$$

Integrating the space interval over all co-moving coordinates to some co-ordinate r, we find the proper distance D:

$$\begin{aligned} D &= \int_0^\tau d\tau && (60)\\ &= \int_0^r R(t)dr && (61)\\ &= R(t)\int_0^r dr && (62)\\ &= R(t)r && (63) \end{aligned}$$

This is simply the relationship between the physical distance and the co-moving coordinate.

Since the co-moving coordinate r does not change in time, we can take the time-derivative straightforwardly, and the remaining steps are identical to those in Method 1 above.

$$\begin{aligned} \dot{D} &= \dot{R}(t)r && (64)\\ v &= \dot{R}(t)r && (65)\\ v &= \dot{R}(t)\frac{D}{R(t)} && (66)\\ v &= \frac{\dot{R}(t)}{R(t)}D && (67)\\ v &= HD && (68) \end{aligned}$$

Appendix C

Details Of Einstein's Cosmological Constant

* Einstein assumed the universe to be *static* $(\dot{R} = 0)$ and *adiabatic* $(dQ = 0)$.

* He used the Friedman (escape) equation to derive a relationship for the *density* in the universe.

* Recall that the Friedman (escape) equation is:

$$\left(\frac{\dot{R}}{R}\right)^2 = \left[\frac{8\pi G}{3}\rho\right] - \left[\frac{kc^2}{R^2}\right] \tag{69}$$

By setting $\dot{R} = 0$, that is, forcing the universe to be static (not changing in time), we can solve for ρ:

$$\rho = \frac{3kc^2}{8\pi G R^2} \tag{70}$$

* With this expression for density, he used the First Law of Thermodynamics and the adiabatic condition to show either density or pressure (due to radiation) must be negative:

$$\rho c^2 = -3p \tag{71}$$

(The calculations involve some elementary calculus and simple algebra. I'll be happy to show anyone who's interested.)

* In the matter dominated era, p = 0, and the above equation would suggest that therefore, ρ also equals zero. Obviously, ρ is *not* zero by our very existence. So, something seemed "wrong".

* Since this was not physically possible, Einstein assumed density and pressure could be rewritten as:

$$\tilde{\rho} = \rho + \frac{\Lambda}{8\pi G} \tag{72}$$

$$\tilde{p} = p - \frac{\Lambda}{8\pi G}c^2 \tag{73}$$

* Using these new expressions for density and pressure in Eq.(71) and assuming we live in a *matter dominated era* (so that p = 0 ... remember, normal p is caused by radiation), we can easily derive after some algebra:

$$\rho = \frac{kc^2}{4\pi G R^2} \tag{74}$$

$$= \frac{\Lambda}{4\pi G} \tag{75}$$

... where

$$\Lambda = kc^2R^2 \tag{76}$$

This new relationship for density allows for a positive quantity of density in a matter dominated era, provided Λ is positive. If Λ is positive, k must be positive. Thus, Einstein's model has a positive geometry, that is, k = +1.

Using this cosmological term Λ, the "corrected" form of the Friedman (escape) equation is:

$$\left(\frac{\dot{R}}{R}\right)^2 = \left[\frac{8\pi G}{3}\rho\right] + \left[\frac{\Lambda}{3} - \frac{kc^2}{R^2}\right] \tag{77}$$

Thus we can see that the gravitational term (the first term in brackets on the right hand side of the equal sign) can now be canceled by the cosmological constant term (the second term in brackets on the right hand side of the equal sign) to give an overall answer of zero, which is needed for a static universe where $\dot{R} = 0$. That is ...

$$\left(\frac{\dot{R}}{R}\right)^2 = \left[\frac{8\pi G}{3}\rho\right] + \left[\frac{\Lambda}{3} - \frac{kc^2}{R^2}\right] \tag{78}$$

$$= \left[\frac{8\pi G}{3}\left(\frac{\Lambda}{4\pi G}\right)\right] + \left[\frac{\Lambda}{3} - \Lambda\right] \tag{79}$$

$$= \left[\frac{2}{3}\Lambda\right] + \left[-\frac{2}{3}\Lambda\right] \tag{80}$$

$$= 0 \tag{81}$$

Without the Λ term, there would be no way to get a static universe solution ($\dot{R}$=0) from the Friedman (escape) equation and at the same time satisfy the First Law of Thermodynamics with the adiabatic condition. Einstein forced the equations to be static by including the fudge factor Λ. Later, when Hubble discovered that the universe was indeed *not* static, Einstein lamented that his fudge factor Λ was his greatest blunder.

Einstein's model was *static*, and Eq.(77) is perfectly fine if R is constant (that is, $\dot{R} = 0$). However, we now know that our universe is actually accelerating. We can still use Einstein's idea of a cosmological constant to express this dynamic universe, and Λ becomes a repulsive force associated with a negative pressure.

However, a universe described by Eq.(77) is *unstable*. To see why, write the Friedman (escape) equation with Λ, k = +1, and $\rho = \rho_0 \left(\frac{R_0}{R}\right)^3$

$$\left(\frac{\dot{R}}{R}\right)^2 = \left[\frac{8\pi G}{3}\rho_0 \left(\frac{R_0}{R}\right)^3\right] + \left[\frac{\Lambda}{3} - \frac{c^2}{R^2}\right] \tag{82}$$

Let's consider an expanding universe. Then, R will increase. The $\left(\frac{1}{R}\right)^3$ gravitational term (which is attractive) will diminish faster than the Λ cosmological constant term (which is repulsive). In fact, as R increases, the sum $\frac{\Lambda}{3} - \frac{c^2}{R^2}$ actually *increases*, because the $\left(\frac{1}{R}\right)^2$ term *decreases*. Thus, the repulsion will dominate as the universe continues to expand. That is, the more the universe expands, the more it wants to continue expanding. Expansion becomes exponential (like the growth of bacteria!), and our universe gets bigger, bigger, and even bigger! This is an *unstable* growth, because there is no bound to the growth. Similarly, a collapsing universe will lead to unstable collapse.

Appendix D

Flat Early Universe

The Friedman (escape) equation is:

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{R^2} \tag{83}$$

$$= \frac{8\pi G}{3}\rho_0 \left(\frac{R_0}{R}\right)^3 - \frac{kc^2}{R^2} \tag{84}$$

... where we used $\rho = \rho_0 \left(\frac{R_0}{R}\right)^3$.

Compare the magnitude of the two terms on the right hand side of the equal sign as we go back in time when the universe is smaller (and so R is smaller).

The gravity term (1st term on the right hand side of the equal sign) is proportional to $\left(\frac{1}{R}\right)^3$.

The curvature (k) term (2nd term on the r.h.s. of the equal sign) is proportional to $\left(\frac{1}{R}\right)^2$.

Thus, as R gets smaller and smaller, the first term will get bigger and bigger *faster* than the second term.

Therefore, when R is very small (very early in the universe), the curvature term is neglible compared to the gravity term. It could effectively be k=0 for all practical purposes! That is, in the early universe, the geometry is basically flat.

A flat universe is associated with $\Omega = 1$.

Thus, in the early universe, the geometry is flat (k = 0) and $\Omega = 1$.